

observe: $|\vec{0}| = |\vec{0}| = 0$
 $|\vec{u}| = 0$ if and only if $\vec{u} = \vec{0}$

Definition: The direction of a vector \vec{u} is the associated unit vector (i.e. vector w/ length 1):

i.e. direction of \vec{u} is $\frac{1}{|\vec{u}|}\vec{u}$ (unit vector) when $\vec{u} \neq \vec{0}$

Claim: $\frac{1}{|\vec{u}|}\vec{u}$ is a unit vector

$$\left| \frac{1}{|\vec{u}|}\vec{u} \right| = \left| \frac{1}{|\vec{u}|} \right| |\vec{u}| = \frac{1}{|\vec{u}|} |\vec{u}| = 1 \quad (\text{when } \vec{u} \neq \vec{0})$$

In \mathbb{R}^3 , there are 3 special vectors, called the component vectors:

$$\begin{aligned} \vec{i} &= \langle 1, 0, 0 \rangle \\ \vec{j} &= \langle 0, 1, 0 \rangle \\ \vec{k} &= \langle 0, 0, 1 \rangle \end{aligned} \quad \left. \vphantom{\begin{aligned} \vec{i} &= \langle 1, 0, 0 \rangle \\ \vec{j} &= \langle 0, 1, 0 \rangle \\ \vec{k} &= \langle 0, 0, 1 \rangle \end{aligned}} \right\} \begin{array}{l} \text{"standard basis"} \\ \text{for } \mathbb{R}^3 \end{array}$$

Every vector is a sum of scalar multiples of component vectors

$$\begin{aligned} \vec{u} &= \langle u_1, u_2, u_3 \rangle \\ &= \langle u_1, 0, 0 \rangle + \langle 0, u_2, 0 \rangle + \langle 0, 0, u_3 \rangle \\ &= u_1 \langle 1, 0, 0 \rangle + u_2 \langle 0, 1, 0 \rangle + u_3 \langle 0, 0, 1 \rangle \end{aligned}$$

8/30/21 12.3 Dot Product (goal: connect algebra of vectors to geometry via a new operation on vectors)

Def: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$

dot product of \vec{u} and \vec{v} is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

(vector · vector → scalar)

$\vec{u} \cdot (\vec{v} \cdot \vec{w})$ doesn't make sense
 - can't have vector · scalar
 $(\vec{u} \cdot \vec{v}) \vec{w}$ makes sense

Ex: $\vec{u} = \langle 1, 3, 5 \rangle$, $\vec{v} = \langle -3, 5, 7 \rangle$

$$\vec{u} \cdot \vec{v} = (1)(-3) + (3)(5) + (5)(7) = -3 + 15 + 35 = 47$$

Theorem: (Properties of Dot Product)Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

$$\textcircled{1} \vec{v} \cdot \vec{v} = v_1 v_1 + v_2 v_2 + v_3 v_3 = v_1^2 + v_2^2 + v_3^2 \\ = (\sqrt{v_1^2 + v_2^2 + v_3^2})^2 = |\vec{v}|^2$$

$$\textcircled{2} \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \vec{v} \cdot \vec{u}$$

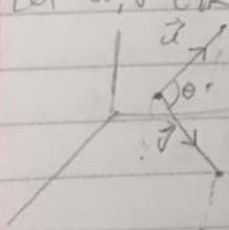
Dot product is commutative (not associative)

$$\textcircled{3} \vec{u} \cdot (\vec{v} + \vec{w}) = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \\ = u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3) \\ = (u_1 v_1 + u_1 w_1) + (u_2 v_2 + u_2 w_2) + (u_3 v_3 + u_3 w_3) \\ = (u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_1 w_1 + u_2 w_2 + u_3 w_3) \\ = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

Dot product distributes over addition

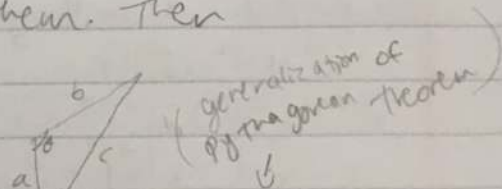
$$\textcircled{4} \vec{u} \cdot (c\vec{v}) = \langle u_1, u_2, u_3 \rangle \cdot \langle cv_1, cv_2, cv_3 \rangle \\ = u_1(cv_1) + u_2(cv_2) + u_3(cv_3) = c(u_1 v_1 + u_2 v_2 + u_3 v_3) \\ = c(\vec{u} \cdot \vec{v})$$

$$\textcircled{5} \vec{0} \cdot \vec{v} = 0$$

Theorem: (geometric interpretation of the dot product)Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and let θ be the angle b/t them. Then

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$$

algebraic \uparrow geometric



Proof: (recall- law of cosines)

$$c^2 = a^2 + b^2 - 2ab \cos(\theta)$$

consider $\vec{u} - \vec{v}$. Applying the law of cosines: $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos \theta$

on the left side we apply properties of Dot Product.

$$|\vec{u} - \vec{v}|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = (\vec{u} - \vec{v}) \cdot \vec{u} - (\vec{u} - \vec{v}) \cdot \vec{v}$$

8/30/21

distributive of dot

algebra

$$\begin{aligned}
 &= (\vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{u}) - (\vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{u} - \vec{u} \cdot \vec{v} \\
 &= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2\vec{u} \cdot \vec{v} \quad (\text{comm. of dot}) \\
 &= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v}
 \end{aligned}$$

$$\begin{aligned}
 |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta \\
 &= |\vec{u} - \vec{v}|^2 \\
 &= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v} \\
 \vec{u} \cdot \vec{v} &= |\vec{u}||\vec{v}|\cos\theta
 \end{aligned}$$

Corollary: Supposing \vec{u} and \vec{v} are both non-zero

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$$

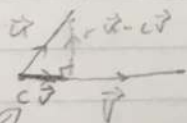
Observation: The zero-vector has an undefined angle with all other vectors

Corollary: If \vec{u} and \vec{v} are perpendicular (i.e. orthogonal), then $\vec{u} \cdot \vec{v} = 0$

conversely - $\vec{u} \cdot \vec{v} = 0$ implies \vec{u} and \vec{v} are orthogonal

Orthogonal Projection

Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$



projection

To project \vec{u} orthogonally onto \vec{v} :

$$\vec{v} \cdot (\vec{u} - c\vec{v}) = 0$$

$$\text{iff } c(\vec{v} \cdot \vec{u}) - c^2(\vec{v} \cdot \vec{v}) = 0$$

$$c(\vec{v} \cdot \vec{u}) - c|\vec{v}|^2 = 0 \quad \text{if either } c=0 \text{ or } \vec{u} \cdot \vec{v} - c|\vec{v}|^2 = 0$$

So Assuming $|\vec{v}| \neq 0$ and $c \neq 0$, $c = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}$

projection of \vec{u} onto \vec{v}
 Def: The orthogonal projection of \vec{u} onto \vec{v} is
 $\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$

$$\text{Proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \left(\frac{1}{|\vec{v}|} \vec{v} \right)$$

unit vector

$$= \text{comp}_{\vec{v}}(\vec{u}) \left(\frac{1}{|\vec{v}|} \vec{v} \right)$$

is the scalar projection of \vec{u} onto \vec{v} is $\text{comp}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$

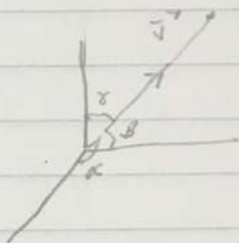
Direction angles: (angles w/ \vec{i} , \vec{j} and \vec{k})

Let $\vec{v} \in \mathbb{R}^3$

The direction angles of \vec{v} are the angles \vec{v} makes \vec{i} , \vec{j} and \vec{k}
 i.e. $\alpha = \arccos \left(\frac{\vec{v} \cdot \vec{i}}{|\vec{v}| |\vec{i}|} \right) = \arccos \left(\frac{v_1}{|\vec{v}|} \right)$

$$\beta = \arccos \left(\frac{v_2}{|\vec{v}|} \right)$$

$$\gamma = \arccos \left(\frac{v_3}{|\vec{v}|} \right)$$



Note: The direction angles determine the "would-be location" of \vec{v} on the unit sphere about the origin.

Exercise: Show that any two of the direction angles of \vec{v} determine the third...